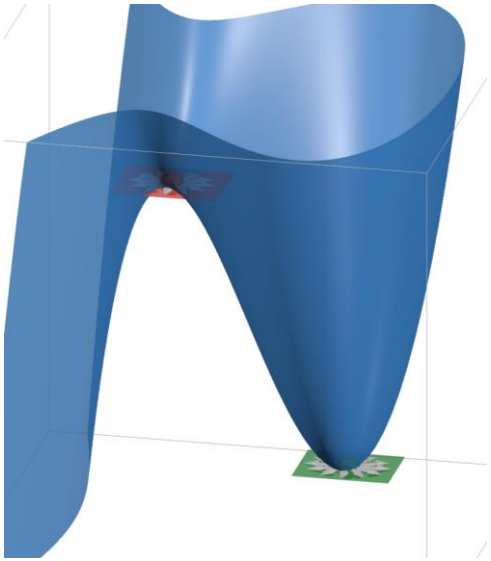
14.7 (part 1) Local Max/Min



Consider the surface z = f(x,y).

A **critical point** is a point (a,b) where **<u>BOTH</u>**

$$f_x(a,b) = 0 \quad \text{AND} \quad f_y(a,b) = 0$$

or where either partial doesn't exist.

Entry Task:

Find the critical points of

$$f(x,y) = x^3 - 2x^2 - 11x + y^2 + 2xy - 2y$$

Second Derivative Test

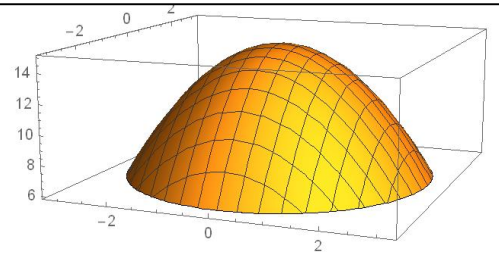
Let (a, b) be a critical point. Find all **second** partials at (a, b) and compute

$$D = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

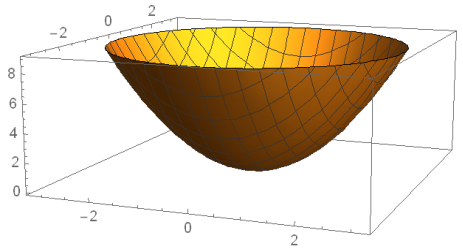
1. If **D > 0**, then concavity same in all directions.
   a. Concave up… **Local Min**
   b. Concave down… **Local Max**
2. If **D < 0**, then concavity changes… **Saddle Point**.
3. If **D = 0**, the test is *inconclusive*.

Quick Examples:

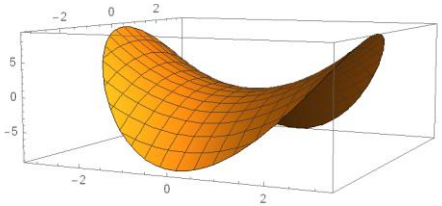
1. $f(x,y) = 15 - x^2 - y^2$,
   A local max occurs at (0,0)



2. $g(x,y) = x^2 + y^2$,
   A local min occurs at (0,0)



3. $h(x,y) = x^2 - y^2$
   A saddle point occurs at (0,0)



Continued Entry Task

Classify the critical points of

$$f(x, y) = x^3 - 2x^2 - 11x + y^2 + 2xy - 2y$$

Critical points: (−1,2) and (3,-2).

*Note: A critical point is only a **candidate** for a local extremum. If D < 0, then it is a saddle point, not a local extrema.*

Visual: https://www.desmos.com/3d/0cikx7zcca

Optional: *Not required for this course. Just for your interest.*

Why Does the 2nd Derivative Test Work?

Recall from 14.6,

for a unit vector $\boldsymbol{u} = \langle h, k \rangle$, $D_{\mathrm{u}} f(a,b) = f_x(a,b)h + f_y(a,b)k$.

At a critical point, (a,b), we have $D_{\mathrm{u}} f(a,b) = 0 \cdot h + 0 \cdot k = 0$ in every direction.

Now consider the second directional derivatives.

$$D_{\mathrm{u}}(D_{\mathrm{u}} f) = D_{\mathrm{u}}(f_{\mathrm{x}} h + f_{\mathrm{y}} k) = (f_{\mathrm{xx}} h + f_{\mathrm{xy}} k)h + (f_{\mathrm{yx}} h + f_{\mathrm{yy}} k)k$$

Assuming $f_{xy} = f_{yx}$, we obtain $D_{\mathrm{u}}{}^2 f = f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2$

Key idea: Look at concavity at (a,b) in the direction of $\boldsymbol{u} = \langle h, k \rangle$?

If $D_{\mathrm{u}}{}^2 f(a,b) > 0$ for all $\boldsymbol{u}$, then it is concave up in ALL directions at (a,b), so (a,b) gives a **local min.**

If $D_{\mathrm{u}}{}^2 f(a,b) < 0$ for all $\boldsymbol{u}$, then it is concave down in ALL directions at (a,b), so (a,b) gives a **local max.**

If concavity changes, then say (a,b) gives a **saddle point.**

It all comes down to studying when $D_{\mathrm{u}}{}^2 f(a,b)$ is positive or negative. Someone cleverly thought to complete the square!

Assuming $f_{\mathrm{xx}} \neq 0$:

$$\begin{aligned} D_{\mathrm{u}}{}^2 f(a,b) &= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \\ &= f_{xx}\left(h + \left(\frac{f_{xy}}{f_{xx}}\right)k\right)^2 + \left(\frac{f_{xx}f_{yy} - (f_{xy})^2}{f_{xx}}\right)k^2 \end{aligned}$$

That is where the mysterious expression D comes from!

If we let $D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$, then we can make the following conclusion…

- If $f_{xx}(a,b) > 0$ and $\frac{D(a,b)}{f_{xx}(a,b)} > 0$, then the surface is concave up in all directions.
  - This happens if $f_{xx} > 0$ and $D > 0$
- If $f_{xx}(a,b) < 0$ and $\frac{D(a,b)}{f_{xx}(a,b)} < 0$, then the surface is concave down in all directions.
  - This happens if $f_{xx} < 0$ and $D > 0$
- If $f_{xx}(a,b)$ and $\frac{D(a,b)}{f_{xx}(a,b)}$ have opposite signs, then the concavity will change (sometimes positive and sometimes negative)
  - This happens if $D < 0$

Technical note: This argument assumes continuous second partial derivatives near the critical point. If D = 0, higher-order behavior may matter.

Examples from old exams:

1. Find and classify all critical points for

$$f(x,y) = x^2 + 4y - x^2y + 1$$

***Visuals**:*
2D - [14.7 Local Max/Min Example 2]
3D - [https://www.desmos.com/3d/dzmothbhny]

2. Find and classify all critical points for

$$f(x,y) = \frac{9}{x} + 3xy - y^2$$

***Visuals**:*
2D - [14.7 Local Max/Min Example 3]
3D - [https://www.desmos.com/3d/xw0qjxfsaj]

14.7 (part 2) Global Max/Min

Recall:

Finding global max/min in one variable

1. Find critical numbers in the interval.
2. Check endpoints.
3. Compare all output values.

Biggest output = absolute max.

Smallest output = absolute min.

Show me you remember this:

Find the absolute (global) maximum and minimum of each function on its interval.

$A)\ f(x) = 2x^2 + 9, \quad 0 \le x \le 2$

$B)\ g(x) = 4x^2 - 4x + 9, \quad 0 \le x \le 2$

$C)\ h(y) = 9, \quad 0 \le y \le 4$

Global Max/Min in Two Variables

Given: $z = f(x, y)$ over a closed region, R, in the xy-plane.

Want: The largest and smallest z-values over R.

How:

Step 1. Draw the region and label its sides.

Step 2. Inside: Find all interior critical pts.

Step 3. Boundary:

On EACH boundary curve, substitute into $f(x, y)$ and solve a one-variable max/min problem.

Step 4. Compare all candidate outputs.

Global Max/Min —Triangular Region

Example (Fall 2018 Exam 2)

Find the absolute maximum and minimum of

$$f(x,y) = 4x^2 - xy + 9$$

over the triangle with vertices

$$(0,0), \qquad (0,4), \qquad (2,4)$$

Step 1. Draw and label the three edges.

Step 2. Check interior critical points.

Step 3. Examine each boundary separately.

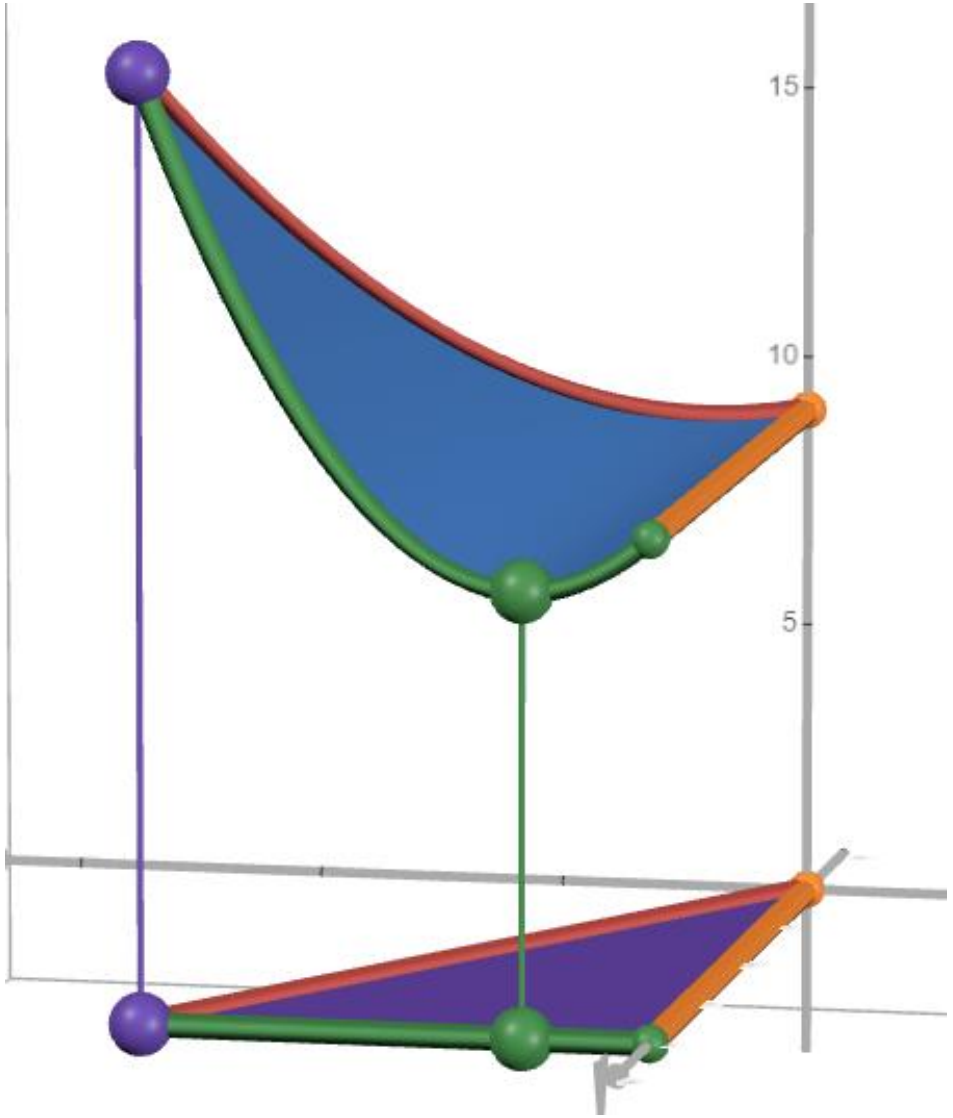
A: $x = 0,\ \ 0 \le y \le 4$

B: $y = 4,\ \ 0 \le x \le 2$

C: $y = 2x,\ \ 0 \le x \le 2$

For each edge: reduce to one variable; find critical numbers and endpoints.

Step 4. Compare the outputs.



Interactive Visuals: https://www.desmos.com/3d/gg9duonj7w

Additional Global Max/Min Practice

Rectangular region

$$f(x,y) = x^2 + y^2 + x^2y + 5$$

$$R = \{(x,y): |x| \leq 1, |y| \leq 1\}$$

Find interior critical points and compare the minimum and maximum on each of the four edges.

Circular region

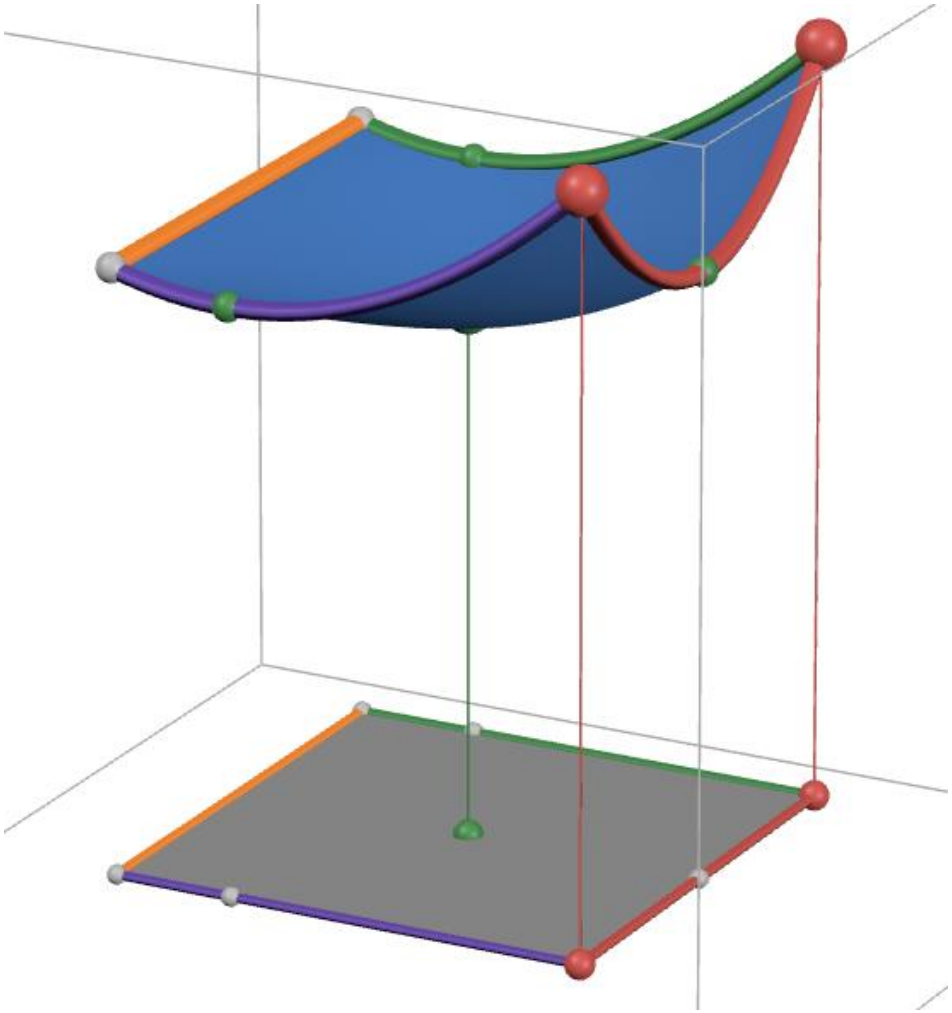
$$f(x,y) = 2x^3 + y^4 + 7$$

$$R = \{(x,y): x^2 + y^2 \leq 1\}$$

How would you handle the circular boundary?

Interactive Visuals:

https://www.desmos.com/3d/ylycappusy



Interactive Visuals:

https://www.desmos.com/3d/yrth2qfdd5

